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LETTERS TO THE EDITORS

On the search for new solutions of the single-pass crossflow heat exchanger problem

IT WAS IN 1954 when Klinkenberg [1], emphasising the justification for renewed discussion of heat transfer in crossflow heat exchangers, wrote:

> It appears that investigators, especially when operating in different fields, have worked along independent lines so that the literature on the subject reveals a great many alternative treatments and duplications.

Now, 30 years later (several more alternative treatments have since been published) we are surprised by yet another duplication of the exact solution to the Nusselt's [2, 3] model of heat transfer in single-pass crossflow heat exchanger. The latter is due to Łach [4]. Solutions of the same problem have been presented elsewhere in somewhat different forms and, in view of the importance of Nusselt's model, the relationship between the various solutions should be recognised. Otherwise, we may see yet another rediscovery in the future.

It is the intention of this letter to point out that the solution of Łach [4] is an alternative and that it is completely equivalent to those of Nusselt [2], Smith [5], Binnie and Poole [6], Mason [7], Kuhl [8], Schedwill [9] and Ishimaru *et al.* [10]. Each of these solutions is explicit and Łach's statement that the "system of equations is usually transformed into a Volterra integral equation" can be regarded as true just in the case of Nusselt's second paper [3] where such a transformation was used.

Several more remarks stand when Łach's paper is considered. The first might be a printing error, however one can note a systematical appearance of an error in the argument of the exponential function in Łach's equations (7) and (9), i.e. $[-a+(ab/s_1+b)]$ should read $-as_1/(s_1+b)$. Secondly, the statement that:

> Mikusinski's operational calculus, used for resolving the problem, is more useful than Laplace transformations because it is easier to proceed from the operator form to the solution to its functional form

does not hold at all. The Laplace transform technique is nothing more than operational calculus and proceeding from the operator form (transformate) to the functional form (original) is straightforward and unique. The Laplace transform has been effectively used to solve the same problem by Mason [7] and Ishimaru *et al.* [10]. Mason's complete procedure is given on pp. 617–621 of Kern and Kraus' book [11] (unfortunately with some obvious printing errors). The Laplace transform method has been proven in many texts as a powerful tool for solving even more general crossflow heat exchanger problems. Just to mention a few recent works. Romie's [12] solution for the transient response of a single-pass crossflow heat exchanger, obtained by Laplace transform, yields a steady-state (Nusselt's) solution as a special case. The solutions for a number of multipass crossflow heat exchanger arrangements presented in [13–15] are results of the application of the Laplace transform technique. Generally speaking, to solve any of the problems mathematically isomorphous with heat transfer in crossflow exchangers, including heat transfer in packed beds and regenerators, mass transfer in adsorption and ion exchange columns, etc., it is sufficient to recognise the following two Laplace transform pairs:

$$V_{i,0}(x,y) \equiv e^{-(x+y)}(y/x)^{(i-1)/2}I_{i-1}(2\sqrt{xy}) = \mathscr{L}^{-1}_{p\to y}\left\{\frac{\exp\left[-xp/(p+1)\right]}{(p+1)^i}\right\} \tag{1}$$

and

$$V_i(x,y) \equiv e^{-(x+y)}\sum_{n=i-1}^{\infty}\binom{n}{i-1}(y/x)^{n/2}I_n(2\sqrt{xy}) = \mathscr{L}^{-1}_{p\to y}\left\{\frac{\exp\left[-xp/(p+1)\right]}{p^i}\right\} \tag{2}$$

where $I_n(\cdot)$ are modified Bessel functions of integer (n) order, and $i = 1, 2, 3, \ldots$. By using equations (1) and (2) together with the principal properties of the Laplace transform, one can solve many problems described by hyperbolic partial differential equations. An example, concerning a three-fluid crossflow heat exchanger, is presented in [16].

The notation $V_{i,0}$ and V_i for the functions with the Laplace transforms given in the equations (1) and (2) is adopted from [17], however in the literature on heat and mass transfer one can find various other notations for the functions of this class. A reference list pleading for completeness might have hundreds of entries and that is why just a few classical and few recent ones will be mentioned here. The $V_1(x,y)$ function, denoted as $J(x,y)$ and named as 'fundamental' by Carslaw and Jaeger [18] is the so-called Anzelius–Schumann [19, 20] function. Its properties have been studied extensively by Goldstein [21] and it is tabulated to six significant figures in Table 10.1 of Sherwood *et al.*'s [22] textbook on mass transfer. Romie [12, 23] denotes the $V_1(x,y)$ function as $H_0(x,y)$, or more generally he uses the $H_{i-1}(x,y)$ functions which are related to the $V_i(x,y)$ functions as $H_{i-1}(x,y) = (i-1)!\,V_i(x,y)/y^{i-1}$. Romie's $B(x,y)$ function is identical to the $V_{1,0}(x,y)$ function, and some other Laplace transform pairs relevant to the problem under consideration are listed in his Table 1 [12].

Since the aim of this letter is to stress the fact that Łach's solution is just an alternative to the already known solutions, we will now show what are in fact the functions $Bes_n(x,y)$ and $Bs_n(x,y)$ used by Łach [4]. The functions

$$Bes_n(x,y) \equiv \sum_{k=\max(-n,0)}^{\infty}\frac{x^{k+n}y^k}{(k+n)!\,k!} \tag{3}$$

used by Łach for $n \geqslant 0$ are nothing more than scaled modified Bessel functions of integer (n) order:

$$Bes_n(x,y) = (x/y)^{n/2}I_n(2\sqrt{xy}) = \sum_{k=n}^{\infty}\frac{y^{k-n}x^k}{(k-n)!\,k!}, \quad \text{for } n \geqslant 0, \tag{4}$$

with double Laplace transform

$$\mathscr{L}_{x\to s}\{\mathscr{L}_{y\to p}\{Bes_n(x,y)\}\} = \frac{1}{s^n(ps-1)}, \quad n \geqslant 0. \tag{5}$$

For $n < 0$ equation (3) states in fact that the arguments x and y should interchange their position in the representation

$$Bes_n(x,y) = (y/x)^{-n/2} I_{-n}(2\sqrt{xy}) = \sum_{k=-n}^{\infty} \frac{x^{k+n}y^k}{(k+n)!k!}, \quad \text{for } n < 0, \tag{6}$$

since $I_{-n} = I_n$, and the Laplace transform is now

$$\mathscr{L}_{x\to s}\{\mathscr{L}_{y\to p}\{Bes_n(x,y)\}\} = \frac{1}{p^{-n}(ps-1)}, \quad \text{for } n < 0. \tag{7}$$

The other class of functions used by Łach [4] are

$$Bs_n(x,y) \equiv \sum_{m=\max(0,n)}^{\infty} \frac{x^m}{m!} \sum_{k=0}^{m-n} \frac{y^k}{k!}. \tag{8}$$

These functions belong to the Neumann series type since, say for $n \geqslant 0$, we have

$$Bs_n(x,y) = \sum_{m=n}^{\infty} \frac{x^m}{m!} \sum_{k=0}^{m-n} \frac{y^k}{k!} = \mathscr{L}_{s\to x}^{-1}\left\{\mathscr{L}_{p\to y}^{-1}\left\{\frac{1}{s^{n-1}(s-1)(ps-1)}\right\}\right\} = \sum_{m=n}^{\infty} (x/y)^{m/2} I_m(2\sqrt{xy}). \tag{9}$$

Łach's [4] final results are expressed in the terms of $Bs_0(x,y)$, $Bs_1(x,y)$ and $Bs_{-1}(x,y)$ functions which are in fact:

$$Bs_0(x,y) = \sum_{m=0}^{\infty} \frac{x^m}{m!} \sum_{k=0}^{m} \frac{y^k}{k!} = \mathscr{L}_{s\to x}^{-1}\left\{\mathscr{L}_{p\to y}^{-1}\left\{\frac{s}{(s-1)(ps-1)}\right\}\right\} = \sum_{m=0}^{\infty} (x/y)^{m/2} I_m(2\sqrt{xy}) = e^{x+y} - \sum_{m=1}^{\infty} (y/x)^{m/2} I_m(2\sqrt{xy}) \tag{10}$$

$$Bs_1(x,y) = \sum_{m=1}^{\infty} \frac{x^m}{m!} \sum_{k=0}^{m-1} \frac{y^k}{k!} = \mathscr{L}_{s\to x}^{-1}\left\{\mathscr{L}_{p\to y}^{-1}\left\{\frac{1}{(s-1)(ps-1)}\right\}\right\} = \sum_{m=1}^{\infty} (x/y)^{m/2} I_m(2\sqrt{xy}) = e^{x+y} - \sum_{m=0}^{\infty} (y/x)^{m/2} I_m(2\sqrt{xy}) \tag{11}$$

$$Bs_{-1}(x,y) = \sum_{m=0}^{\infty} \frac{x^m}{m!} \sum_{k=0}^{m+1} \frac{y^k}{k!} = \mathscr{L}_{s\to x}^{-1}\left\{\mathscr{L}_{p\to y}^{-1}\left\{\frac{s(p+1)-1}{p(s-1)(ps-1)}\right\}\right\} = e^{x+y} - \sum_{m=2}^{\infty} (y/x)^{m/2} I_m(2\sqrt{y/x}) = \sqrt{y/x}\, I_1(2\sqrt{xy}) + \sum_{m=0}^{\infty} (x/y)^{m/2} I_m(2\sqrt{xy}). \tag{12}$$

The last equality in the equations (10)–(12) is related to the fundamental expansion theorem of the Bessel coefficients and the generating function of the modified Bessel functions (see Goldstein [21, p. 159]).

Recognising $Bs_0(x,y)$ as in equation (10) above, one finds it identical to the function θ of Binnie and Poole's [6] equation (30). Furthermore, the final form of the Łach's solution [4, equation (19)] is exactly the same as the equations (17) and (18) of Binnie and Poole [6].

Before we proceed with the comparison of various other available results let us demonstrate the use of the Laplace transform technique for solving the single-pass crossflow heat exchanger problem with arbitrary inlet fluid temperature distributions. We will use coefficient-free equations and widely accepted heat exchanger parameters [24]: $N_{tu} = kA/C_{min}$ is the number of transfer units and $C^* = C_{min}/C_{max}$ is the fluid capacity rate ratio. If we denote all quantities referring to the weaker fluid (C_{min}) flowing in ξ direction with subscript 1 and those referring to stronger fluid (C_{max}) flowing in η direction with subscript 2, then the dimensionless fluid temperature fields $\theta_1(\xi,\eta)$ and $\theta_2(\xi,\eta)$ in a core of a single-pass crossflow heat exchanger are governed by the following equations

$$\frac{\partial\theta_1}{\partial\xi} + \theta_1 - \theta_2 = 0 \tag{13}$$

$$\frac{\partial\theta_2}{\partial\eta} + \theta_2 - \theta_1 = 0 \tag{14}$$

in $0 \leqslant \xi \leqslant N_{tu}$ and $0 \leqslant \eta \leqslant C^*N_{tu}$

subject to the inlet conditions

$$\theta_1(0,\eta) = \phi_1(\eta) \tag{15}$$

$$\theta_2(\xi,0) = \phi_2(\xi). \tag{16}$$

When the solution to the above problem is available one can determine the separating wall temperature distribution from the following equation

$$\theta_w(\xi,\eta) = \frac{S\theta_1(\xi,\eta)+\theta_2(\xi,\eta)}{S+1}, \quad \text{with} \quad S = (\eta_0\bar{\alpha}A)_1/(\eta_0\bar{\alpha}A)_2 \tag{17}$$

where η_0, $\bar{\alpha}$ and A are the extended surface efficiency, average heat transfer coefficient and total heat transfer area at the respective sides of the wall.

It is convenient to apply the Laplace transform with respect to both variables, i.e. $\eta \to p$ and $\xi \to s$, in order to solve the system (13)–(16). The application of the Laplace transform

$$\theta_{iL} \equiv \theta_{iL}(\xi,p) = \mathscr{L}_{\eta\to p}\{\theta_i(\xi,\eta)\}, \quad i = 1,2 \tag{18}$$

to the equations (13) and (15) yields

$$\frac{d\theta_{1L}}{d\xi} + \theta_{1L} - \theta_{2L} = 0 \tag{19}$$

and

$$\theta_{1L}(0,p) = \phi_{1L}(p) \tag{20}$$

respectively, while from the equation (14), with (16) taken into account, one obtains

$$(p+1)\theta_{2L} - \theta_{1L} = \phi_2(\xi). \tag{21}$$

Now, applying the transform

$$\tilde{\theta}_{iL} \equiv \tilde{\theta}_{iL}(s,p) = \mathscr{L}_{\xi\to s}\{\theta_{iL}(\xi,p)\}, \quad i = 1,2 \tag{22}$$

to the equations (19)–(21) one obtains a set of algebraic equations

$$(s+1)\tilde{\theta}_{1L} - \tilde{\theta}_{2L} = \phi_{1L}(p) \tag{23}$$

$$(p+1)\tilde{\theta}_{2L} - \tilde{\theta}_{1L} = \tilde{\phi}_2(s), \tag{24}$$

from which one obtains

$$\tilde{\theta}_{1L} = \frac{(p+1)\phi_{1L}(p)}{ps+s+p} + \frac{\tilde{\phi}_2(s)}{ps+s+p} \tag{25}$$

$$\tilde{\theta}_{2L} = \frac{\phi_{1L}(p)}{ps+s+p} + \frac{(s+1)\tilde{\phi}_2(s)}{ps+s+p}. \tag{26}$$

These results can be rewritten as

$$\tilde{\theta}_{1L} = \frac{\phi_{1L}(p)}{s+\dfrac{p}{p+1}} + \frac{\tilde{\phi}_2(s)}{(s+1)\left(p+\dfrac{s}{s+1}\right)} \tag{27}$$

$$\tilde{\theta}_{2L} = \frac{\phi_{1L}(p)}{(p+1)\left(s+\dfrac{p}{p+1}\right)} + \frac{\tilde{\phi}_2(s)}{p+\dfrac{s}{s+1}}. \tag{28}$$

In order to obtain temperature fields $\theta_i(\xi,\eta)$, $i = 1,2$, from the equations (27) and (28) one has to apply the inversion theorem. For the first terms on the RHS of these equations it is plausible to apply first the transformation $\mathscr{L}^{-1}_{s\to\xi}$ and then $\mathscr{L}^{-1}_{p\to\eta}$, while for the second terms this order should be reversed, i.e.

$$\theta_1(\xi,\eta) = \mathscr{L}^{-1}_{p\to\eta}\left\{\mathscr{L}^{-1}_{s\to\xi}\left\{\frac{\phi_{1L}(p)}{s+\dfrac{p}{p+1}}\right\}\right\} + \mathscr{L}^{-1}_{s\to\xi}\left\{\mathscr{L}^{-1}_{p\to\eta}\left\{\frac{\tilde{\phi}_2(s)}{(s+1)\left(p+\dfrac{s}{s+1}\right)}\right\}\right\} \tag{29}$$

$$\theta_2(\xi,\eta) = \mathscr{L}^{-1}_{p\to\eta}\left\{\mathscr{L}^{-1}_{s\to\xi}\left\{\frac{\phi_{1L}(p)}{(p+1)\left(s+\dfrac{p}{p+1}\right)}\right\}\right\} + \mathscr{L}^{-1}_{s\to\xi}\left\{\mathscr{L}^{-1}_{p\to\eta}\left\{\frac{\tilde{\phi}_2(s)}{p+\dfrac{s}{s+1}}\right\}\right\}. \tag{30}$$

When the first inversion is performed one is left with the equations

$$\theta_1(\xi,\eta) = \mathscr{L}^{-1}_{p\to\eta}\left\{\phi_{1L}(p)\exp\left(-\frac{\xi p}{p+1}\right)\right\} + \mathscr{L}^{-1}_{s\to\xi}\left\{\tilde{\phi}_2(s)\frac{\exp\left(-\dfrac{\eta s}{s+1}\right)}{s+1}\right\} \tag{31}$$

$$\theta_2(\xi,\eta) = \mathscr{L}^{-1}_{p\to\eta}\left\{\phi_{1L}(p)\frac{\exp\left(-\dfrac{\xi p}{p+1}\right)}{p+1}\right\} + \mathscr{L}^{-1}_{s\to\xi}\left\{\tilde{\phi}_2(s)\exp\left(-\frac{\eta s}{s+1}\right)\right\} \tag{32}$$

and it is now obvious that the second inverse transformation will yield the convolution of the analytical functions with the prescribed functions at the boundaries [inlet temperature distributions $\phi_1(\eta)$ and $\phi_2(\xi)$, respectively]:

$$\theta_1(\xi,\eta) = \int_0^\eta \phi_1(v)[e^{-\xi}\delta(\eta-v)+V_{2,0}(\eta-v,\xi)]\,dv + \int_0^\xi \phi_2(u)V_{1,0}(\eta,\xi-u)\,du = \phi_1(\eta)\,e^{-\xi} + \int_0^\eta \phi_1(v)V_{2,0}(\eta-v,\xi)\,dv$$
$$+ \int_0^\xi \phi_2(u)V_{1,0}(\eta,\xi-u)\,du \tag{33}$$

$$\theta_2(\xi,\eta) = \int_0^\eta \phi_1(v)V_{1,0}(\xi,\eta-v)\,dv + \phi_2(\xi)\,e^{-\eta} + \int_0^\xi \phi_2(u)V_{2,0}(\xi-u,\eta)\,du. \tag{34}$$

Łach's [4] equations (8) and (9) are completely equivalent to these general results obtained by the Laplace transforms.

To include a wide range of practical situations let the arbitrary inlet fluid temperature distributions be presented by a power series or polynomials of the form

$$\phi_1(\eta) = \sum_{n=0}^{N_1} \beta_{1n}\eta^n/n! = \sum_{n=0}^{N_1} \beta_{1n}\mathscr{L}^{-1}_{p\to\eta}(1/p^{n+1}), \quad 0 \leqslant N_1 \leqslant \infty \tag{35}$$

$$\phi_2(\xi) = \sum_{n=0}^{N_2} \beta_{2n}\xi^n/n! = \sum_{n=0}^{N_2} \beta_{2n}\mathscr{L}^{-1}_{s\to\xi}(1/s^{n+1}), \quad 0 \leqslant N_2 \leqslant \infty. \tag{36}$$

Then, from equations (31) and (32) we have

$$\theta_1(\xi,\eta) = \sum_{n=0}^{N_1} \beta_{1n}\mathscr{L}^{-1}_{p\to\eta}\left\{\frac{\exp\left(-\frac{\xi p}{p+1}\right)}{p^{n+1}}\right\} + \sum_{n=0}^{N_2} \beta_{2n}\mathscr{L}^{-1}_{s\to\xi}\left\{\frac{\exp\left(-\frac{\eta s}{s+1}\right)}{s^{n+1}(s+1)}\right\} = \sum_{n=0}^{N_1} \beta_{1n}V_{n+1}(\xi,\eta) + \sum_{n=0}^{N_2} \beta_{2n}\sum_{m=n+2}^{\infty}(-1)^{m+n}V_m(\eta,\xi) \quad (37)$$

$$\theta_2(\xi,\eta) = \sum_{n=0}^{N_1} \beta_{1n}\mathscr{L}^{-1}_{p\to\eta}\left\{\frac{\exp\left(-\frac{\xi p}{p+1}\right)}{p^{n+1}(p+1)}\right\} + \sum_{n=0}^{N_2} \beta_{2n}\mathscr{L}^{-1}_{s\to\xi}\left\{\frac{\exp\left(-\frac{\eta s}{s+1}\right)}{s^{n+1}}\right\} = \sum_{n=0}^{N_1} \beta_{1n}\sum_{m=n+2}^{\infty}(-1)^{m+n}V_m(\xi,\eta) + \sum_{n=0}^{N_2} \beta_{2n}V_{n+1}(\eta,\xi). \quad (38)$$

In these equations the identity $1/[z^M(z+1)] = \sum_{m=M+1}^{\infty}(-1)^{m+M+1}(z)^{-m}$ ($Re\ p > 1, M \geqslant 1$) has been used once for $z = s$ and once for $z = p$.

One can note the reversed order of arguments ξ and η in the functions multiplying the coefficients β_{1n} and β_{2n}. The functions $V_i(x, y)$ are not symmetrical with respect to the interchange of the position of their arguments, and for $i \geqslant 2$ the following relation holds

$$V_i(y,x) = \frac{x^{i-1}}{(i-1)!} + \sum_{j=2}^{i} \frac{(-1)^{j-1}x^{i-j}}{(i-j)!}\sum_{k=0}^{j-2}\binom{j-2}{k}\frac{y^{j-k-1}}{(j-k-1)!} + (-1)^i\sum_{l=0}^{i-2}\binom{i-2}{l}V_{i-l}(x,y). \quad (39)$$

That this is true can be seen immediately by taking the double Laplace transform of both sides of this equation. For $i = 1$, see Goldstein [21, equation (47)], we have

$$V_1(y,x) = 1 + V_{1,0}(x,y) - V_1(x,y). \quad (40)$$

From the general solution, equations (37) and (38), for arbitrary inlet temperature distributions, it is easy to obtain particular solutions for uniform inlet temperatures—the case studied by many authors as mentioned above—by putting $\phi_1(\eta)$ and $\phi_2(\xi)$ to be some constants, say ϕ_1^c and ϕ_2^c, respectively. Thus, one should use $\beta_{1n} = \delta_{n0}\phi_1^c$ and $\beta_{2n} = \delta_{n0}\phi_2^c$, where δ_{n0} is Kronecker's delta, in equations (37) and (38) to obtain

$$\theta_1(\xi,\eta) = \phi_1^c V_1(\xi,\eta) + \phi_2^c\sum_{m=2}^{\infty}(-1)^m V_m(\eta,\xi) \quad (41)$$

$$\theta_2(\xi,\eta) = \phi_1^c\sum_{m=2}^{\infty}(-1)^m V_m(\xi,\eta) + \phi_2^c V_1(\eta,\xi). \quad (42)$$

However,

$$\sum_{m=2}^{\infty}(-1)^m V_m(x,y) = V_1(x,y) - V_{1,0}(x,y) = 1 - V_1(y,x) \quad (43)$$

so that finally

$$\theta_1(\xi,\eta) = \phi_2^c + (\phi_1^c - \phi_2^c)V_1(\xi,\eta) \quad (44)$$

$$\theta_2(\xi,\eta) = \phi_1^c - (\phi_1^c - \phi_2^c)V_1(\eta,\xi). \quad (45)$$

If the crossflow heat exchanger is an element of the heat exchanger network, then the inlet temperatures ϕ_1^c and ϕ_2^c will correspond to the values at particular gridpoints. However, if the single-pass unit is regarded *per se*, then it is convenient to choose the dimensionless weaker fluid inlet temperature to be equal to one ($\phi_1^c = 1$) and that of stronger fluid to be zero ($\phi_2^c = 0$). Then, from (44) and (45) the well-known solution is

$$\theta_1(\xi,\eta) = V_1(\xi,\eta) \quad (46)$$

$$\theta_2(\xi,\eta) = 1 - V_1(\eta,\xi). \quad (47)$$

It is just the V_1 function's different representations that made the solutions of various authors appear different. To recognise them as completely equivalent we note that following identities hold:

$$V_1(x,y) = 1 - e^{-y}\int_0^x e^{-u}I_0(2\sqrt{yu})\,du \quad (48)$$

$$= e^{-(x+y)}I_0(2\sqrt{xy}) + e^{-x}\int_0^y e^{-v}I_0(2\sqrt{xv})\,dv \quad (49)$$

$$= e^{-(x+y)}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{y^n x^k}{n!k!} \quad (50)$$

$$= 1 - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+k}\frac{(k+n)!}{(k+1)!k!(n!)^2}x^{k+1}y^n \quad (51)$$

$$= e^{-x}\left[1 + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}(-1)^m\frac{(m+n-1)!}{n!m!(n-1)!}x^n\frac{y^{m+n}}{(m+n)!}\right] \quad (52)$$

$$= 1 - e^{-y}\left[\sum_{m=0}^{\infty}(-1)^m\frac{x^{m+1}}{(m+1)!} + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}(-1)^m\frac{(m+n)!}{m!(n!)^2}y^n\frac{x^{m+n+1}}{(m+n+1)!}\right] \quad (53)$$

$$= e^{-(x+y)} \sum_{n=0}^{\infty} (y/x)^{n/2} I_n(2\sqrt{xy}) \quad (54)$$

$$= 1 - e^{-(x+y)} \sum_{n=1}^{\infty} (x/y)^{n/2} I_n(2\sqrt{xy}). \quad (55)$$

The simplest way to prove any of these equations is by taking the Laplace transform with respect to both variables and using some additional algebra. The choice of one or the other relation is governed by purely computational aspects, i.e. the convergence criteria, which we will not discuss here. We just note that the preference of some formula strongly depends on the relative magnitude of the arguments x and y, and for some asymptotic relations the reader should consult the papers by Goldstein [21] and Klinkenberg [1].

With the identities (48)–(55) available, one can make a unified survey of all available solutions to the problem under consideration. In Table 1 we present the rediscovery of Nusselt's solutions by eight other authors. To verify their complete equivalence one should apply some of the relations given by equations (48)–(55), the 'interchange of the arguments' relation (40), and to recall that $J_n(iz) = i^n I_n(z)$. For example, the use of the latter property of Bessel functions of imaginary argument makes the Schedwill [9] equations (4.70) and (4.71) immediately identical to the Binnie and Poole [6] equations (17) and (18), respectively.

Having the solution, i.e. the explicit relations for fluid temperature fields, $\theta_1(\xi,\eta)$ and $\theta_2(\xi,\eta)$, one can obtain any additionally desired information in a straightforward manner. Let us summarise some of these results.

The local driving force in the exchanger, i.e. the local fluid temperature difference, is [from (46) and (47) with (40) and (1) being used]:

$$\theta_1(\xi,\eta) - \theta_2(\xi,\eta) = V_{1,0}(\xi,\eta) = e^{-(\xi+\eta)} I_0(2\sqrt{\xi\eta}) \quad (56)$$

as given for the first time by Binnie and Poole [6] in their equation (19), and rediscovered later by Mason [7] in his equation (11).

The wall temperature distribution may be simply obtained by introducing (46) and (47) into (17) as

$$\theta_w(\xi,\eta) = V_1(\xi,\eta) - \frac{V_{1,0}(\xi,\eta)}{S+1}, \quad (57)$$

and was not considered by any of the authors. It is worthwhile to make a quick analysis of this result. An obvious conclusion is that the wall temperature $\theta_w(\xi,\eta)$ coincides with the weaker fluid temperature distribution $\theta_1(\xi,\eta)$ for $S \to \infty$ (negligible resistance on side 1), and with stronger fluid temperature $\theta_2(\xi,\eta)$ for $S \to 0$ (negligible resistance on side 2). For any finite value of the resistance ratio S the wall temperature will be between $\theta_1(\xi,\eta)$ and $\theta_2(\xi,\eta)$ as illustrated in Fig. 1 for symmetric resistances ($S = 1$).

Table 1. Unified survey of the solutions to the crossflow heat exchanger problem*

Author	Ref.	Solution	Equation
Nusselt, 1911	[2]	$T/T_0 = V_1(KFy/Wy_0, KFx/wx_0)$ $t/T_0 = 1 - V_1(KFx/wx_0, KFy/Wy_0)$	11, 11a 12
Nusselt, 1930	[3]	$T(x,y) = V_1(by, ax)$ $\Theta(x,y) = 1 - V_1(ax, by)$	19, 20 13b, 19, 20
Smith, 1934†	[5]	$\frac{t-t_1'}{t_1-t_1'} = V_1(y', x')$	see the third column on p. 606
Binnie and Poole, 1937	[6]	$F = V_1(p,q)$ $f = 1 - V_1(q,p)$	17 18
Mason, 1955‡	[7]	$G(x,y) = V_{1,0}(ay, bx)$	11
Kühl, 1959	[8]	$v_0 = V_1(x,y)$ $\Theta_0 = 1 - V_1(y,x)$	22 23
Schedwill, 1968	[9]	$\frac{v-v_1'}{v_1-v_1'} = V_1(ay, bx)$ $\frac{v'-v_1'}{v_1-v_1'} = 1 - V_1(bx, ay)$	4.70 4.71
Ishimaru *et al.*, 1976	[10]	$\theta_h(X,Y) = V_1(NTU_h Y, NTU_c X)$ $\theta_c(X,Y) = 1 - V_1(NTU_c X, NTU_h Y)$	9 (1) 9 (2)
Romie, 1983	[12]	$\tau_{a\infty} = V_1\left(\frac{XR}{1+R}, \frac{Y}{1+R}\right)$ $\tau_{b\infty} = 1 - V_1\left(\frac{Y}{1+R}, \frac{XR}{1+R}\right)$	13 14
Łach, 1983	[4]	$\Theta_1(\xi,\eta) = V_1(a\xi, b\eta)$ $\Theta_2(\xi,\eta) = 1 - V_1(b\eta, a\xi)$	19 (1) 19 (1)

* The notations of dependent and independent variables are those from the corresponding reference.

† Smith's formula on p. 606 contains the following printing errors: the first summation should start from $u = 0$, and the factorial sign with v in the numerator should be omitted.

‡ Mason presents in fact the result for dimensionless local fluid temperature difference.

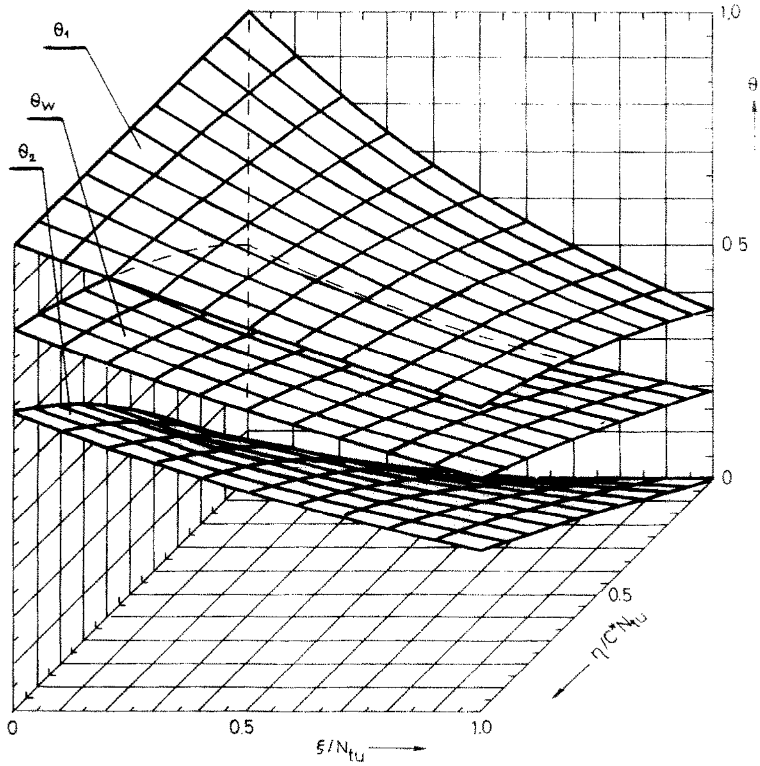

FIG. 1. Fluids and wall temperature fields in a crossflow heat exchanger.

An interesting result, perhaps, is along the 'diagonal' $z = \xi = \eta$ of the separating wall. The wall temperatures at such locations are

$$\theta_w(z,z) = \frac{1}{2}\left[1+\frac{S-1}{S+1}e^{-2z}I_0(2z)\right] \tag{58}$$

and it is clear that the wall will be isothermal (at $\theta_w = \frac{1}{2}$, the arithmetic mean of the fluids inlet temperatures) along this line if the resistances are symmetric ($S = 1$) on both its sides. The location of the 'diagonal' is strongly dependent on the fluid capacity rate ratio (C^*) as presented in Fig. 2, and the opposing thermal stresses in the wall's regions bounded respectively by this line have proved to be crucial in practice.

The outlet fluid temperature profiles

$$\theta_1(N_{tu}, \eta) = V_1(N_{tu}, \eta) \tag{59}$$

and

$$\theta_2(\xi, C^*N_{tu}) = 1 - V_1(C^*N_{tu}, \xi) \tag{60}$$

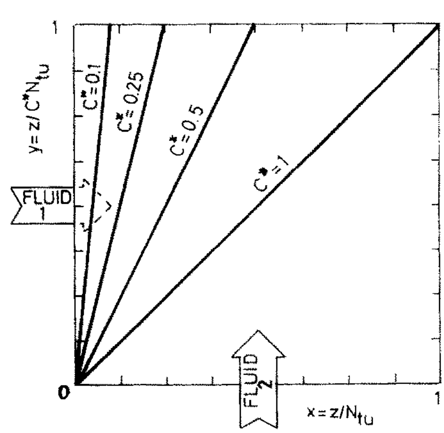

FIG. 2. The lines along which the exchanger core becomes isothermal for the same resistances on both sides.

can be used for the evaluation of the mean outlet temperatures

$$\theta_{1,\text{out}} = \frac{1}{C^*N_{\text{tu}}} \int_0^{C^*N_{\text{tu}}} \theta_1(N_{\text{tu}}, \eta)\,\mathrm{d}\eta \tag{61}$$

and

$$\theta_{2,\text{out}} = \frac{1}{N_{\text{tu}}} \int_0^{N_{\text{tu}}} \theta_2(\xi, C^*N_{\text{tu}})\,\mathrm{d}\xi, \tag{62}$$

respectively. When this is done one obtains

$$\theta_{1,\text{out}} = V_2(N_{\text{tu}}, C^*N_{\text{tu}})/C^*N_{\text{tu}} \tag{63}$$

$$\theta_{2,\text{out}} = 1 - V_2(C^*N_{\text{tu}}, N_{\text{tu}})/N_{\text{tu}} \tag{64}$$

where the V_2 function belongs to the same class defined by equation (2) and is simply the integral of the V_1 function with respect to the second argument. The overall balance equation for the exchanger

$$\theta_{2.\text{out}} = C^*(1-\theta_{1.\text{out}}) \tag{65}$$

is satisfied by the last two equations since the interchange of the position of arguments in V_2 function [see (39) for $i = 2$] obeys the relation

$$V_2(y,x) = x - y + V_2(x,y). \tag{66}$$

The heat exchanger effectiveness

$$\varepsilon = 1 - \theta_{1,\text{out}} = \theta_{2,\text{out}}/C^* \tag{67}$$

for Nusselt's crossflow assumptions, will thus be

$$\varepsilon = \varepsilon(C^*, N_{\text{tu}}) = 1 - V_2(N_{\text{tu}}, C^*N_{\text{tu}})/C^*N_{\text{tu}} \tag{68}$$

and one can arrive at as many different, but equivalent, formulae for $\varepsilon(C^*, N_{\text{tu}})$ as there are possible ways to present the V_2 function. It is immediately clear that, since

$$V_2(x,y) = \int_0^y V_1(x,u)\,\mathrm{d}u, \tag{69}$$

and since there is a long list [see (48)–(55)] of expressions for the V_1 functiôn, one can expect a number of rediscoveries of formula (68). This has happened several times during more than half a century.

Reviewing the results of various authors, again in chronological order, one can derive the following conclusions on the search for the effectiveness formula.

In his pioneering work [2] Nusselt did not use any term that resembles the heat exchanger effectiveness, but he did point out the way of calculating the actual heat transfer in his equations (19) and (19a). Since the effectiveness is merely the dimensionless (normalised to the values between 0 and 1) actual heat transfer rate, one can deduce, by making Nusselt's equation (19) dimensionless and using his equation (18) following from (11) (see the first entry in our Table 1 as well), that the quantity

$$\frac{Q}{WT_0} = 1 - \frac{1}{x_0}\int_0^{x_0} V_1(KF/W, KFx/wx_0)\,\mathrm{d}x = 1 - \frac{w}{KF}\int_0^{KF/w} V_1(KF/W, u)\,\mathrm{d}u \tag{70}$$

in his notation stands for the effectiveness. Upon recognising that Nusselt's KF/W stands for N_{tu} and KF/w for C^*N_{tu}, one easily verifies that equation (70) leads, via (69), to equation (68).

The search for new solutions to the crossflow heat exchanger problem was initiated by Nusselt himself in his second paper [3]. He was aware of the highly transcendental nature of the solution and thus started studies which have continued to the present day. In 1930 Nusselt realised the advantage of the use of dimensionless mean fluid outlet temperatures and we can follow his steps in terms of the V_i functions as follows. His equations (21) and (22) represent $V_1(b, ax)$, so that equation (23) can be read as

$$T_m = \int_0^1 V_1(b, ax)\,\mathrm{d}x, \tag{71}$$

which yields (23a), such that, together with (24), is just a representation of the V_2 function, i.e.

$$T_m = V_2(b,a)/a. \tag{72}$$

When he later, in the first of his equations (26), introduces the quantity $\xi = 1 - T_m$ we can interpret this as the effectiveness for the cases $a \leqslant b$, i.e. $a = C^*N_{\text{tu}}$ and $b = N_{\text{tu}}$. If $a > b$ then Nusselt's quantity denoted by η is the effectiveness and ξ becomes $C^*\varepsilon$ where C^* stands for his b/a. In other words Nusselt was able to show that the aforementioned relations (63) and (64) combined with (67) verify the unique solution.

Finally Nusselt was the first to write down, in the third of his equations (26), the relation that we now recognise to be valid for any heat exchanger: the effectiveness divided by the number of transfer units is a dimensionless mean temperature difference:

$$\Delta\theta_m = \frac{\varepsilon}{N_{\text{tu}}} = \frac{1}{C^*N_{\text{tu}}^2}\int_0^{C^*N_{\text{tu}}}\int_0^{N_{\text{tu}}} [\theta_1(\xi,\eta) - \theta_2(\xi,\eta)]\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{73}$$

In 1934 Smith [5] presents an implicit formula for the dimensionless mean temperature difference denoting it by r. We can easily establish that his formula for the effectiveness is

$$\varepsilon = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} (-1)^{u+v} \frac{(u+v)!}{u!(u+1)!v!(v+1)!} C^{*v} N_{\text{tu}}^{u+v+1} \tag{74}$$

since he uses the variables $r = \Delta\theta_m = \varepsilon/N_{\text{tu}}$, $p = \varepsilon$ and $q = C^*\varepsilon$. To prove the validity of Smith's formula, one has just to combine the representation of V_1 function given by (51) with equations (68) and (69).

The effectiveness formula of Binnie and Poole [6], following from their equation (21) with $q = C^*N_{tu}$ and $p = N_{tu}$, is

$$\varepsilon = 1 - \frac{1}{C^*N_{tu}} e^{-N_{tu}(1+C^*)} \sum_{n=1}^{\infty} nC^{*n/2} I_n(2N_{tu}\sqrt{C^*}) \tag{75}$$

which is again identical to (68) when $V_2(N_{tu}, C^*N_{tu})$ is introduced from equation (2).

Mason [7] derived the effectiveness formula in the following form

$$\varepsilon = \frac{1}{C^*N_{tu}} \sum_{n=0}^{\infty} \left[1 - e^{-N_{tu}} \sum_{k=0}^{n} (N_{tu})^k/k!\right]\left[1 - e^{-C^*N_{tu}} \sum_{m=0}^{n} (C^*N_{tu})^k/k!\right] \tag{76}$$

by evaluating the dimensionless mean temperature difference and using the relation (73). This is again completely equivalent to equation (68).

With Kühl's [8] notation $\eta_0 = \varepsilon$, $x_0 = N_{tu}$ and $y_0 = C^*N_{tu}$, one can recognise that his equations (27) and (28) combine to give

$$\varepsilon = 1 - \frac{1}{C^*N_{tu}} e^{-N_{tu}(1+C^*)} \sum_{n=0}^{\infty} (N_{tu})^n/n! \sum_{m=n+1}^{\infty} (m-n)(C^*N_{tu})^m/m! \tag{77}$$

which is another representation of the result (68) with V_2 obtained by integrating by parts (69) with integrand of the form given in (50). We note that the Kühl's formula for effectiveness can be written as

$$\varepsilon(C^*, N_{tu}) = 1 - e^{-N_{tu}} - e^{-N_{tu}(1+C^*)} \sum_{n=1}^{\infty} (N_{tu})^n/n! \sum_{m=n+1}^{\infty} (m-n)(C^*N_{tu})^{m-1}/m! \tag{78}$$

or, upon rearranging the order of double summation, as

$$\varepsilon(C^*, N_{tu}) = 1 - e^{-N_{tu}} - e^{-N_{tu}(1+C^*)} \sum_{n=1}^{\infty} (C^*N_{tu})^n/(n+1)! \sum_{m=1}^{n} (n+1-m)(N_{tu})^m/m!. \tag{79}$$

Either of these forms is suitable for recognition that $\varepsilon(0, N_{tu}) = 1 - \exp(-N_{tu})$ is valid for the Nusselt's crossflow heat exchanger model as it is for any other flow arrangement.

Schedwill's [9] equation (4.84) with $\phi = \varepsilon$, $kF/W_1 = N_{tu}$ and $W_1/W_2 = C^*$ is absolutely identical to Binnie and Poole's result presented above in equation (75). One has just to express the Bessel functions of imaginary argument as modified Bessel functions.

Ishimura *et al.* [10] presented the effectiveness [their equation (10)] as

$$\varepsilon = 1 - e^{-NTU_c}\left\{1 + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (-1)^m \frac{(m+n-1)!}{n!m!(n-1)!} (NTU_c)^{m+2n} \frac{R^{m+n}}{(m+n+1)!}\right\} \tag{80}$$

where $R = W_c/W_h$ — the heat capacity rate ratio is simply the ratio of the hot side and cold side number of transfer units ($R = NTU_h/NTU_c$). Incidently, in the last term of this expression there is an obvious error in [10], since instead of $(m+n+1)!$ just $(m+n)!$ is printed. Their result is obtained by using V_1, given in (52), integrating it as in (69) and equation (68) is again reproduced for $NTU_c = N_{tu}$ and $NTU_h = C^*N_{tu}$.

Another representation of the result for $\varepsilon(C^*, N_{tu})$ given by (68) is presented in [25]. It is in the form of a Neumann series similar to the Binnie and Poole formula, equation (75), but slightly rearranged, with the help of a recursive relation for modified Bessel functions, into an advantageous relation so that the result for a balanced exchanger ($C^* = 1$):

$$\varepsilon = 1 - e^{-2N_{tu}}[I_0(2N_{tu}) + I_1(2N_{tu})] \tag{81}$$

becomes immediately recognisable.

Just before the recent appearance of Łach's paper [4], Romie [12] proved, by his equation (20), that the result (68) for the effectiveness is obtainable as a special (steady-state) case of a more general (transient) consideration of crossflow heat exchanger problem.

And finally we arrive at Łach's [4] rediscovery of the effectiveness formula. For $\phi_k = \varepsilon$, $a = N_{tu}$ and $b = C^*N_{tu}$ one can read his equation (29) as

$$\varepsilon = \frac{1}{C^*} + e^{-N_{tu}(1+C^*)}\left[Bs_1(N_{tu}, C^*N_{tu}) - \frac{1}{C^*} Bs_{-1}(N_{tu}, C^*N_{tu})\right]. \tag{82}$$

However, Bs_1 and Bs_{-1} functions have the representations given by the equations (11) and (12) and if one chooses the forms starting with the term e^{x+y}, Łach's result becomes

$$\varepsilon = 1 - e^{-N_{tu}(1+C^*)}\left[I_0(2N_{tu}\sqrt{C^*}) + \sqrt{C^*}I_1(2N_{tu}\sqrt{C^*}) + \frac{C^*-1}{C^*} \sum_{m=2}^{\infty} (C^*)^{m/2} I_m(2N_{tu}\sqrt{C^*})\right] \tag{83}$$

which is absolutely identical to the formula presented in [25].

Łach [4] plotted the ε–N_{tu} curves in his Fig. 2, but it appears that the graph is not as accurate as many others previously published (see for example [9, 11, 24, 26, 27]). Tables of numerical values of crossflow heat exchanger effectiveness have also appeared in various texts, but unfortunately, not always with the same adequate accuracy. These are often referred to in connection with various comparisons and it is worthwhile, for archival purposes, to have reliable ε-values as those given in Table 2 where the six decimal places shown are just truncations from ε-values calculated to 12 significant figures. The CPU time for generating this table was 1 s on the Amdahl V/7 computer at Leeds University.

Before ending this comment let us stress that Łach's [4] statement in the fourth paragraph of his conclusions should be regarded in two ways. The first sentence, "The exact solution of the problem has made it possible to derive the formulas for basic cross-flow recuperator parameters", is equally applicable to the work of all his predecessors and not only can the formulae already discussed be derived from the exact solution (whatever its form is), but some additional information as well. This will be discussed below. Łach's second statement, of the same paragraph, that "it is not necessary to perform calculations using formulas for the counter-flow recuperator and the appropriate correction factors", might be misleading. Namely, it is clear that the explicit relations for $\theta_{1,out}$, $\theta_{2,out}$, ε, $\Delta\theta_m$ are *all* given as the functions of N_{tu} and C^*, but the problem of presenting $N_{tu} = f(\varepsilon, C^*)$ or $\Delta\theta_m = \varphi(\varepsilon, C^*)$ for the Nusselt model of crossflow heat exchanger appears to be intractable, as correctly stated by Binnie and

Table 2. Effectiveness $\varepsilon = \varepsilon(C^*, N_{tu})$ of a single-pass crossflow heat exchanger with neither fluid mixed

N_{tu} \ C^*	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.094712	0.094263	0.093818	0.093376	0.092936	0.092499	0.092065	0.091633	0.091204	0.090778
0.2	0.179642	0.173033	0.176444	0.174874	0.173322	0.171789	0.170274	0.168776	0.167297	0.165835
0.3	0.255876	0.252626	0.249431	0.246289	0.243200	0.240162	0.237175	0.234238	0.231349	0.228508
0.4	0.324374	0.319180	0.314095	0.309117	0.304243	0.299471	0.294798	0.290222	0.285741	0.281352
0.5	0.385982	0.378678	0.371555	0.364606	0.357827	0.351214	0.344762	0.338467	0.332324	0.326330
0.6	0.441447	0.431974	0.422761	0.413802	0.405088	0.396613	0.388370	0.380351	0.372550	0.364961
0.7	0.491431	0.479805	0.468526	0.457584	0.446967	0.436666	0.426671	0.416973	0.407562	0.398430
0.8	0.536520	0.522816	0.509544	0.496691	0.484244	0.472190	0.460517	0.449214	0.438267	0.427666
0.9	0.577233	0.561564	0.546407	0.531748	0.517570	0.503859	0.490601	0.477782	0.465386	0.453402
1.0	0.614031	0.596538	0.579628	0.563284	0.547490	0.532230	0.517490	0.503252	0.489501	0.476222
1.1	0.647320	0.628164	0.609644	0.591749	0.574461	0.557766	0.541649	0.526092	0.511079	0.496595
1.2	0.677466	0.656812	0.636836	0.617525	0.598869	0.580853	0.563464	0.546685	0.530501	0.514897
1.3	0.704789	0.682810	0.661529	0.640941	0.621039	0.601814	0.583255	0.565348	0.548079	0.531434
1.4	0.729578	0.706443	0.684007	0.662276	0.641248	0.620922	0.601289	0.582342	0.564069	0.546457
1.5	0.752087	0.727962	0.704518	0.681771	0.659732	0.638405	0.617791	0.597886	0.578683	0.560173
1.6	0.772546	0.747589	0.723276	0.699635	0.676692	0.654460	0.632948	0.612161	0.592098	0.572753
1.7	0.791156	0.765519	0.740467	0.716048	0.692300	0.669251	0.646922	0.625323	0.604462	0.584339
1.8	0.808101	0.781925	0.756256	0.731165	0.706706	0.682923	0.659848	0.637502	0.615901	0.595052
1.9	0.823543	0.796957	0.770786	0.745122	0.720039	0.695597	0.671842	0.648808	0.626520	0.604992
2.0	0.837627	0.810753	0.784184	0.758037	0.732409	0.707378	0.683005	0.659337	0.636410	0.614247
2.5	0.891788	0.864935	0.837658	0.810218	0.782842	0.755718	0.729004	0.702829	0.677296	0.652487
3.0	0.926771	0.901573	0.875068	0.847659	0.819708	0.791528	0.763389	0.735516	0.708100	0.681291
3.5	0.949776	0.927040	0.902123	0.875530	0.847734	0.819163	0.790197	0.761165	0.732346	0.703973
4.0	0.965148	0.945154	0.922217	0.896880	0.869637	0.841156	0.811766	0.781942	0.752057	0.722426
4.5	0.975567	0.958288	0.937463	0.913608	0.887289	0.859082	0.829544	0.799191	0.768482	0.737816
5.0	0.982718	0.967964	0.949236	0.926957	0.901668	0.873972	0.844482	0.813790	0.782438	0.750904
5.5	0.987680	0.975191	0.958459	0.937768	0.913594	0.886531	0.857229	0.826340	0.794482	0.762212
6.0	0.991157	0.980649	0.965771	0.946632	0.923610	0.897260	0.868246	0.837268	0.805012	0.772109
6.5	0.993614	0.984813	0.971628	0.953978	0.932111	0.906523	0.877870	0.846887	0.814318	0.780867
7.0	0.995364	0.988016	0.976360	0.960119	0.939392	0.914594	0.886356	0.855432	0.822619	0.788687
7.5	0.996618	0.990498	0.980214	0.965292	0.945677	0.921682	0.893896	0.863083	0.830081	0.795726
8.0	0.997522	0.992434	0.983372	0.969679	0.951138	0.927948	0.900642	0.869980	0.836837	0.802106
8.5	0.998177	0.993952	0.985975	0.973422	0.955912	0.933521	0.906713	0.876235	0.842990	0.807924
9.0	0.998655	0.995149	0.988133	0.976632	0.960108	0.938503	0.912207	0.881939	0.843623	0.813257
9.5	0.999004	0.996097	0.989929	0.979398	0.963812	0.942930	0.917201	0.887164	0.853805	0.818169
10.0	0.999260	0.996851	0.991430	0.981791	0.967096	0.947018	0.921761	0.891972	0.858593	0.822713

Poole [6] almost 50 years ago. Thus the, practically very important, sizing problem when a designer starts with four known temperatures at the exchanger inlets and outlets in order to determine necessary heat transfer area for required duty, will lead always to the procedures that involve some kind of trial and error steps. At that stage either correction factors, or graphical methods on ε–N_{tu} charts, or interpolation methods based on tabulated functions, or computerised search techniques, or approximate formulae will be required.

Finally, let us briefly review some of the information that various authors *have not* extracted from Nusselt's solution.

Having the solution (46) and (47) to the problem one can calculate the mean temperatures of *each* fluid in heat exchanger core:

$$\theta_{1m} = \frac{1}{C^*N_{tu}^2}\int_0^{C^*N_{tu}}\int_0^{N_{tu}} \theta_1(\xi,\eta)\,d\xi\,d\eta \tag{84}$$

and

$$\theta_{2m} = \frac{1}{C^*N_{tu}^2}\int_0^{C^*N_{tu}}\int_0^{N_{tu}} \theta_2(\xi,\eta)\,d\xi\,d\eta. \tag{85}$$

These might be then used for the more precise evaluation of the thermophysical properties of the fluids, which are assumed to be constant in Nusselt's model.† For a proper choice of the conductivities of the wall and fin materials it is necessary to have the information on the mean wall temperature in the exchanger. Since the local wall temperature is governed by the local fluid temperatures and the ratio of heat transfer resistances of each side, according to equation (17), the same type of relation will be valid for the mean wall and fluid temperatures:

$$\theta_{wm} = \frac{S\theta_{1m}+\theta_{2m}}{S+1}. \tag{86}$$

It must be clear that we need to evaluate just one of the mean integral temperatures, say θ_{2m}, since the other will be

$$\theta_{1m} = \theta_{2m}+\Delta\theta_m \tag{87}$$

where $\Delta\theta_m = \varepsilon/N_{tu}$ [see equation (73)]. The true mean temperature of the 'stronger' fluid, that undergoes the smaller changes in temperatures, is

$$\theta_{2m} = \frac{1}{C^*N_{tu}^2}\int_0^{C^*N_{tu}}\int_0^{N_{tu}} [1-V_1(\eta,\xi)]\,d\xi\,d\eta = \frac{C^*}{2} - \frac{V_3(N_{tu}, C^*N_{tu})}{C^*N_{tu}^2} \tag{88}$$

where V_3 function is defined by equation (2) as well. For $C^* \to 0$ (condenser or evaporator case) $\theta_{2m} = 0$ and $\theta_{1m} = (1-e^{-N_{tu}})/N_{tu}$ since then $\varepsilon = 1-e^{-N_{tu}}$ for any exchanger. Another limit of interest is for $N_{tu} \to \infty$ when $\varepsilon \to 1$ for any C^*: both θ_{2m} and θ_{1m} are then equal to $C^*/2$ which is the arithmetic mean as well. Finally, for the balanced exchanger ($C^* = 1$) we have

$$\theta_{2m} = (1-\Delta\theta_m)/2 = (1-\varepsilon_1/N_{tu})/2 \tag{89}$$

$$\theta_{1m} = (1+\Delta\theta_m)/2 = (1+\varepsilon_1/N_{tu})/2 \tag{90}$$

$$\theta_{wm} = \left(1+\frac{S-1}{S+1}\Delta\theta_m\right)\Big/2 = \left(1+\frac{S-1}{S+1}\frac{\varepsilon_1}{N_{tu}}\right)\Big/2 \tag{91}$$

where ε_1 stands for the effectiveness given by equation (81).

The evaluation of true mean temperatures is readily accommodated on the computer, but to facilitate the hand calculations in the design procedure, it is convenient to have a joint $\varepsilon-N_{tu}$ and $\theta_{2m}-N_{tu}$ diagram as presented in Fig. 3. For an exchanger operating at point A with N_{tuA}, C^*_A, ε_A one can find on the lower part of the graph θ_{2mA} on the same abscissa N_{tuA} and for same value of C^*_A. A straight line through the origin and point A will cut the value of $\Delta\theta_m$ as the vertical at $N_{tu} = 1$. The other mean temperature θ_{1m} is then simply the sum of $\Delta\theta_m$ and θ_{2m} as shown schematically in Fig. 3. Note that the reciprocal value of the true mean temperature difference can be read at the N_{tu} axis at the location where the OA line cuts $\varepsilon = 1$.

Of many other pieces of information which can be extracted from Nusselt's solution by analytical procedures, let us mention just another one that might be useful in various optimisation tasks for crossflow heat exchanger. We have in mind the local slope of the $\varepsilon-N_{tu}$ curve:

$$\left(\frac{\partial\varepsilon}{\partial N_{tu}}\right)_{C^*} = \frac{I_1(2N_{tu}\sqrt{C^*})}{N_{tu}\sqrt{C^*}}\exp\left[-N_{tu}(1+C^*)\right]. \tag{92}$$

This is easily obtained from the equation (68). This function will appear in many optimisation problems. Namely, whatsoever is the optimisation criterion $K = K(C^*, N_{tu}, X_i, \ldots)$, the extremum condition $\partial K/\partial X_i = 0$, where X_i is some design variable to be optimised, will always contain, due to chain rule differentiation [for example $(\partial K/\partial\varepsilon)(\partial\varepsilon/\partial N_{tu})(\partial N_{tu}/\partial X_i) = 0$], the factor $\partial\varepsilon/\partial N_{tu}$ which represents the rate of the effectiveness increment with increasing the number of transfer units, i.e. the heat transfer area.

This result and many others discussed above are in closed form and are readily obtained so we may conclude that the crossflow heat exchanger problem modelled on the Nusselt assumptions is completely exhausted and that no 'new' solutions are necessary or needed. However, there is a certain probability that some other journal in some other time will publish a new rediscovery of the same solution in an other notation.

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† Common practice [28] is to use mean arithmetic temperatures for both fluids if $C^* \geqslant 0.5$, and just for stronger fluid if $C^* < 0.5$ when for the weaker fluid one should add the exchanger's log-mean temperature to the arithmetic mean temperature of the stronger fluid. This can provide satisfactory results for very large (in terms of N_{tu}) exchangers when the performances are practically insensitive to the relatively small changes in exchanger size. The arithmetic means can and should be used as orientational values up to the design procedure instance when N_{tu} becomes known. Starting from that moment the design procedure should incorporate the true mean temperatures and the influence of exchanger's real size will thus be correctly accommodated.

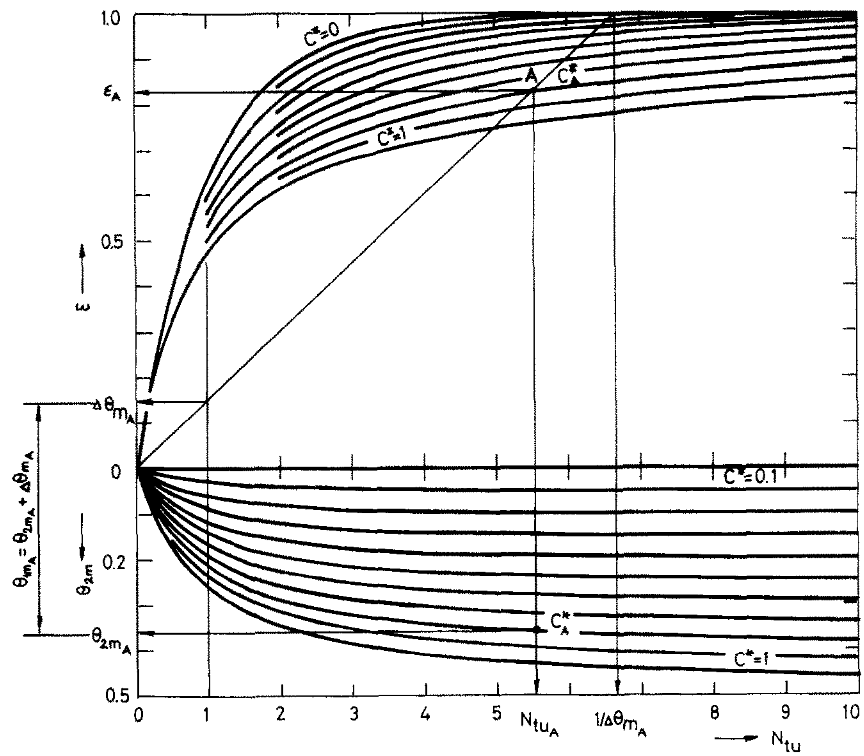

FIG. 3. A joint plot of the effectiveness and mean integral temperature of the stronger fluid as a function of N_{tu} and C^*.

Say, how about the new formula:

$$\varepsilon = 1 - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+n} \frac{(n+m)!(N_{tu})^n}{m!(n!)^2} \left[\frac{(C^* N_{tu})^m}{(m+1)!} + \frac{(C^* N_{tu})^{m+1}}{(m+2)!} \right] ?$$

Well, this is probably the longest letter to the Editors of the *International Journal of Heat and Mass Transfer* ever written. We sincerely believe that Wilhelm Nusselt, from whom we still learn, deserves the time we spent and the space you, the Editors, provided for (hopefully) the closure of the vicious circle around one of his earliest heat transfer problems.

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On the general solution to a certain class of heat and/or mass transfer problems

WE WOULD like to thank B. S. Baclic and P. J. Heggs for very valuable remarks presented in their letter to the Editors [1] on the paper [2]. The first part of their letter is concerned with the problems which we found necessary to respond to by presenting our common view. We hope that such a discussion will prove helpful in clearing up, and perhaps partly, in unifying the approach to a certain class of problems in application of heat and/or mass transfer theory. Most of the remarks presented in [1] had already been discussed in our private communication after [2] and [3] and before [4] were published. We are thus not surprised by the letter of Baclic and Heggs. It was our intention to make a similar review concerned with the methods applied to solutions of the problems postulated in the theory of heat and/or mass transfer, which are described by partial differential equations of the first degree as well as by Volterra's integral equations. So far only a small part of the project has been realized [5] and [6] in which the results achieved by Nusselt, Carslaw and Jaeger, Anzelius, Schumann (see refs. [2, 3, 18–20] of ref. [1]), Thomas [7], Brinkley [8] and Bonilla *et al.* [9] were discussed. More works within this scope are being advanced. These also include analysis of some references given in [1]. We acknowledge the fact that Baclic and Heggs quote the additional publications which should also be discussed.

The review of *various equivalent forms* of the solution of the Nusselt's model of the crossflow recuperator [1] is very valuable. We thank the authors for the effort they have made. A diversified approach to the foregoing problems (including the crossflow recuperator) stems from numerous trials in defining some special functions related to the Bessel functions. As far as we know, the authors of the special functions quoted in the letter did not deal with mathematical aspects of the boundary problem presented below. These authors, however, defined some special functions based only on the analysis of particular cases of the boundary problem. As to their Table 1, it should be supplemented at least by the works of Rabinovitch, whose results are presented in [10]. We should admit that we were not familiar with all the forms of the solution given by Baclic and Heggs, on the other hand one should mention here considerations by di Federico [11] which are also concerned with the problem discussed. However, the latter takes into account heat generation in one of the liquids. In [11] the set of Volterra's integral equations is solved.

With regard to the minor remark made by Baclic and Heggs in their letter, it is worth noting here that under some assumptions differential equations are frequently replaced by integral equations, which in such a case are just Volterra's equations. Next, one can then apply different versions of the fixed point theorem, of which those of Banach and Schauder are most commonly used. Such a treatment makes it possible to analyze both linear and nonlinear equations unlike these where operational calculus (including Laplace's transformation) are applied.

Before we go into more specific discussion we should admit that indeed in [2] there is a printing error in the argument of the exponential functions which was noticed in [1].

Further considerations will be refered to broader problems defined in the analyses of recuperators, regenerators, ion-exchange columns, chemical reactors, nuclear reactors etc. These problems are frequently undertaken anew, e.g. by Montakhab [12], and this has prompted us to elaborate [13].